Deterministic Response to Free-Surface Waves

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The relative importance and method of application of the impulse response technique for predicting motions, forces, etc., due to free-surface wave effects on water-borne vehicles, has been discussed by several authors. Controversy exists over the applicability of the law of causality when there is more than one independent variable; that is, the free surface and the body whose response is desired are distributed in space and vary in time. It is shown in this paper that impulse response operators for systems with free-surface wave inputs, in principle, do not satisfy the law of causality. However, by properly positioning the wave input point, the contribution from negative time can be made small.

Nomenclature

g	= gravitational constant
$h(t)$	= time history of surface elevation at origin
$p(x, y, t)$	= pressure field
t	= time
x, y	= orthogonal Cartesian coordinate system
ζ	= complex Fourier transform parameter, $\zeta = \omega + i\sigma$
$\eta(x, t)$	= free-surface elevation
ρ	= fluid density
$\varphi(x, y, t)$	= velocity potential

EFFECTIVE operation of marine vehicles in the unsteady ocean environment requires analytical methods to predict motions, accelerations, deck wetness, forefoot emergence, etc. Two equivalent approaches have been described for such predictions. St. Denis and Pierson[1] presented methods for determining the probabilistic description of the behavior of a vehicle in a random seaway. The complementary procedure presented by Cummins[2] and exploited by Wehausen[3] yields the deterministic evaluation of a vehicle's reponse to a specified time history of the sea surface. This approach was extended by Henry[4] to the prediction of the rigid body and elastic response of hydrofoil craft. Either method has advantages in attaining effective designs for marine vehicles. This paper is concerned with the deterministic approach and, in particular, with the requirement that the impulse response functions satisfy the law of causality, which states that no physical system can respond to future causes, i.e., the cause must always precede the effect.

The deterministic approach to calculating response of physical systems has been highly developed in the field of control theory[5] which deals with systems described by ordinary differential equations, i.e., with inputs and outputs functions of one independent variable, time. For stable, linear, time-invariant systems, it is well known that the output $y(t)$ of a so-called dynamical system can be predicted from the previous history of the input $\eta(\tau)$, $\tau < t$ by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} \eta(\tau) W(t - \tau)\, d\tau$$

where $W(t)$ is the response of the system to a unit impulsive input at $t = 0$. The foregoing relation then superimposes the responses to all previous elementary input impulses of strength $\eta(\tau)\, d\tau$.

For such systems, $W(t)$ must be zero for $t < 0$; i.e., the system cannot know that the impulsive disturbance is coming. However, this condition, no matter how logical, cannot be proven or concluded from physical or mathematical evidence when treating more complicated systems. Thus, the law of causality is an axiom in the mathematical development and is not a derived fact.

In applying the deterministic approach to the transient responses of vehicles to free-surface waves, it is desired to extend the previous convolution integral to predict the output $y(t)$ that describes the state of the vehicle, from the previous history of the free-surface elevation $\eta(x_1,t)$. In developing this extension, some confusion has arisen concerning the applicability of the law of causality to physical systems that are disturbed in space and are also time-dependent.

Results of many previous studies of transient free-surface waves do satisfy the law of causality. For example, the Cauchy-Poisson problem[6] gives the response of the free-surface and velocity potential to an initial displacement and zero velocity distributed over the entire surface. More recently, Miles[7] has considered the transient wave response to nonstationary pressures on the free surface, whereas Finklestein[8] has treated wave generation by nonstationary pressures on the free surface and by transient motions of submerged bodies in water of finite depth. Furthermore, Ursell[9] has included transient free-surface effects in deriving heaving motions of a semisubmerged, two-dimensional cylinder released from rest or responding to an external force.

In all the preceding cases, details of the disturbance creating the waves are known either explicitly or implicitly through an equation of motion. On the other hand, when a vehicle encounters a free-surface wave system, the cause of the incident wave system is not known in explicit detail. Therefore, in recent studies of vehicle response, the actual input or disturbing element (the wind at sea or a wave generator in a towing tank) has been replaced by a measurement of the free-surface elevation at one point.

Using this approach, Fuchs[10] derived an impulse response function that did not vanish for negative time when treating the response of linear systems to wave inputs. The results were used by Fuchs and MacCamy[11] to express the time histories of heaving and pitching of a ship model in terms of the recorded wave motion. To predict the motion at present time t, knowledge of the future wave motion is required, and, thus, the law of causality is not satisfied. Further credence was given to this position by several authors who used experimentally determined impulse response functions that did not satisfy the causality condition and obtained predictions of vehicle motions which agreed with measurements.[12–14] It consequently was argued by Breslin et al.[15] that impulse response operators for vehicle response to free-surface waves need not satisfy the law of causality.

In arriving at this conclusion, it is argued that a dynamical system subjected to a free-surface wave responds to pressures from all waves over the entire free surface. Thus, waves that

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have not yet reached the input terminal (the wave height indicator) are already affecting the pressure field at the system. This argument is, of course, correct, and in principle the dynamical system is affected by the whole free-surface wave system at the present time. The law of causality stipulates that the dynamical system cannot be affected by the free-surface waves at any time in the future. To illustrate, suppose waves are generated by a wavemaker at the end of a semi-infinite channel and that a wave indicator is placed at some distance down the channel and a dynamical system responding to the waves placed farther down the channel. The wavemaker is allowed to run for some time. At each instant, the wave indicator, in principle, senses the effects of the wavemaker (the cause) from the start up to the present time, and the system simultaneously responds to all these effects. The instant that the wavemaker is turned off, the wave indicator and the dynamical system are, in principle, simultaneously affected by the stopping transient, as are the waves just in front of the wave indicator. Thus, as these waves pass the indicator, they have already been affected by the stopping transient. If knowledge of these waves is required in order to predict the dynamical system response prior to the stopping of the wavemaker, then the response of the system includes a priori knowledge of the wavemaker stopping transient. By the same reasoning, a priori knowledge of the starting transient is also required. It is intuitively obvious that the dynamical system cannot have prior knowledge of the wavemaker transients since these actions are entirely independent of the system. If this were not the case, then no model testing in waves could be carried out. By reductio ad absurdum, the premise of this line of reasoning is incorrect, and, thus, any dynamical system with free-surface wave inputs must satisfy the law of causality.

Having thus concluded that the law of causality must be satisfied, consider a simple example of a distributed system with a free-surface wave input: given the free-surface elevation at $x = 0$ in a two-dimensional channel because of some disturbance in $x \ll 0$, describe the free-surface wave system at $x > 0$, presuming no waves are generated in $x > 0$.

For irrotational flow of an incompressible inviscid fluid in a two-dimensional channel of infinite depth the velocity potential $\varphi(x,y,t)$ is required to satisfy[6] (with y measured vertically upwards, and x horizontal),

$$\varphi_{xx} + \varphi_{yy} = 0 \tag{1}$$

$$\varphi_{tt} + g\varphi_y = 0; \quad y = 0 \tag{2}$$

The free-surface elevation $\eta(x,t)$ is related to φ by

$$\eta(0,t) = (1/g)\varphi_t(x,0,t) \tag{3}$$

and the free-surface elevation at the origin is prescribed by

$$\eta(0,t) = (1/g)\varphi_t(0,0,t) = h(t) \tag{4}$$

It is required to find $\eta(x,t)$ and $\varphi(x,y,t)$ in terms of $h(t)$.

The preceding boundary-value problem for $\varphi(x,y,t)$ can be solved by means of Fourier transforms on t, defined by

$$\bar{\varphi}(x,y,\zeta) = \int_{-\infty}^{\infty} \varphi(x,y,t)e^{i\zeta t}\,dt$$

$$\bar{\eta}(x,\zeta) = \int_{-\infty}^{\infty} \eta(x,t)e^{i\zeta t}\,dt, \quad \bar{h}(\zeta) = \int_{-\infty}^{\infty} h(t)e^{i\zeta t}\,dt \tag{5}$$

where $\zeta = \omega + i\sigma$, and necessary conditions for the existence of the transforms are assumed.[16] The transformed Eqs. (1) and (2) become

$$\bar{\varphi}_{xx} + \bar{\varphi}_{yy} = 0; \quad -\zeta^2\bar{\varphi} + g\bar{\varphi}_y = 0, \quad y = 0$$

which has the solution

$$\bar{\varphi}(x,y,\zeta) = A\exp[\zeta^2(y + ix)/g] + B\exp[\zeta^2(y - ix)/g] \tag{6}$$

In the time domain, it is required that no disturbances are generated in $x > 0$. This can be interpreted in the ζ plane by noting from the inverse Fourier transform

$$\varphi(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\varphi}(x,y,\zeta)e^{-i\zeta t}d\zeta; \quad \zeta = \omega + i\sigma_0$$

that for real ζ, $\varphi(x,y,t)$ is the summation of many progressive wave components of amplitudes $|\bar{\varphi}|$. These waves must only be traveling toward $+x$. In order that Eq. (6) satisfy this condition, take

$$B = 0; \quad \omega > 0 \qquad A = 0; \quad \omega < 0$$

Then,

$$\bar{\varphi}(x,y,\zeta) = \begin{cases} A\exp[\zeta^2(y + ix)/g]; & \omega > 0 \\ B\exp[\zeta^2(y - ix)/g]; & \omega < 0 \end{cases}$$

Finally, the transform of Eq. (4) leads to

$$A = B = (ig/\zeta)\bar{h}(\zeta)$$

and thus

$$\bar{\varphi}(x,y,\zeta) = (ig/\zeta)\bar{h}(\zeta)e^{z\zeta^2} \tag{7}$$

where

$$z = \frac{y \pm ix}{g}; \quad \begin{cases} \text{upper sign in } Re(\zeta) > 0 \\ \text{lower sign in } Re(\zeta) < 0 \end{cases} \tag{8}$$

If the transform of the impulse response function is defined as

$$\overline{W}(x,y,\zeta) = e^{z\zeta^2} \tag{9}$$

then Eq. (7) becomes

$$\bar{\varphi}(x,y,\zeta) = (ig/\zeta)\bar{h}(\zeta)\overline{W}(x,y,\zeta) \tag{10}$$

The inverse transform of this result can be written as

$$\varphi(x,y,t) = g\int_{-\infty}^{\infty} d\tau\, W(x,y,\tau)\int_{-\infty}^{t-\tau} h(\lambda)d\lambda \tag{11}$$

The wave height in $x > 0$ is then obtained by introducing Eq. (11) into (3), which gives

$$\eta(x,t) = \int_{-\infty}^{\infty} W(x,0,\tau)\,h(t - \tau)\,d\tau \tag{12}$$

where the impulse response function is obtained as the inverse Fourier transform of Eq. (9).

The question now arises as to whether or not the impulse response operator $W(x,y,t)$ satisfies the law of causality; i.e., does it vanish for all $t < 0$? It can be shown that a function $f(t)$ with transform $\bar{f}(\zeta)$ vanishes for $t < 0$ if, and only if, $\bar{f}(\zeta)$ satisfies the condition[16]

$$\lim_{|\zeta| \to \infty} |\bar{f}(\zeta)| = 0 \quad \text{in } Im(\zeta) > 0 \tag{13}$$

Letting $\zeta = Re^{i\theta}$ and $(y \pm ix)/g = \rho e^{\pm i\beta}$, then Eqs. (8) and (9) yield

$$\overline{W}(x,y,\zeta) = \exp[\rho R^2\cos(2\theta \pm \beta)]; \quad \begin{cases} \text{upper sign in } 0 < \theta < \pi/2 \\ \text{lower sign in } \pi/2 < \theta < \pi \end{cases} \tag{14}$$

It is then found that $\overline{W}(x,y,\zeta)$ satisfies Eq. (13) only when

$$0 < \theta < 3\pi/4 - \beta/2 \text{ or } \pi/4 + \beta/2 < \theta < \pi$$

and that $|W(x,y,\zeta)|$ is exponentially divergent in the sector of the ζ plane defined by

$$3\pi/4 - \beta/2 < \theta < \pi/4 + \beta/2$$

From its definition, the angle β is

$$\beta = \tan^{-1}(x/y); \quad \pi/2 < \beta < \pi$$

and thus when $y = 0$ and $x > 0$, $\beta = \pi/2$ and the divergent sector vanishes. Thus, $W(x,0,\zeta)$ satisfies Eq. (13), but $W(x,y,\zeta)$, $y < 0$, does not. Consequently, the impulse response operator in Eq. (12) vanishes for $\tau < 0$ so that the wave height in $x > 0$ is given by

$$\eta(x,t) = \int_0^\infty W(x,0,\tau)\, h(t - \tau)\, d\tau \quad (15)$$

whereas the impulse response operator for the velocity potential in $y < 0$ in Eq (11) does not vanish for $\tau < 0$.

These impulse response functions have been determined previously. Davis and Zarnick[12] have shown that

$$W(x,0,t) = a \cos[(\pi/2)a^2t^2]\, [\tfrac{1}{2} + C(at)] + a \sin[(\pi/2)a^2t^2]\, [\tfrac{1}{2} + S(at)] \quad (16)$$

which is correct for $t > 0$, whereas for $t < 0$ it is shown here that $W(x,0,t)$ vanishes. Previous studies[12, 15] have incorrectly used Eq. (16) for all t. In the case $y < 0$, Breslin et al.[15] have obtained

$$W(x,y,t) = \tfrac{1}{2}\left(\frac{g}{\pi}\right)^{1/2} Re\left\{\frac{e^{a/(y - ix)}}{(-y + ix)^{1/2}} \times \left[1 + \text{Erf}\, i\left(\frac{a}{-y + ix}\right)^{1/2}\right]\right\} \quad (17)$$

where $a = gt^2/4$.

It was shown previously that the response of a system to a free-surface wave input must satisfy the law of causality, but the impulse response operator given in Eq. (17) does not. This apparent paradox can be explained as follows. In the mathematical statement of the preceding boundary-value problem, the boundary conditions are given at only one point on the boundary $y = 0$. However, it is known that this is not sufficient to guarantee a unique solution to Laplace's equation. Thus, the conditions not given on $x < 0$, $h = 0$ are replaced in the solution by future conditions at $x = 0$.

To avoid the need for future information to predict the present state of a system involving the free surface, either the cause of the free-surface disturbance must be known in detail or the wave height over the whole free surface at the present time must be known. Neither of these appears to be practically attainable. Thus, for convenience it would be advantageous to use the wave height at one point as the input signal. In fact, there is evidence that although the impulse response operator does not vanish, its contribution for $t < 0$ can be made small. When $x = 0$ and $y < 0$, Breslin et al.[15] found that the impulse function was symmetrical in time, while as x increases, the contribution from $t < 0$ diminishes. Furthermore, it was shown previously that the impulse response operator vanishes for $t < 0$ when $y = 0$, $x < 0$. These observations suggest that there may be a sector $\pi/2 < \beta < \beta_0$ in which the contribution of $W(x,y,t)$ for $t < 0$ can be sensibly neglected. It is tentatively concluded, therefore, that by properly positioning the wave height input point, the actual impulse response operator for a system with a free-surface wave as input can be approximated by an impulse response function that does vanish for negative time.

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